Dynamic Response of an Elastic Plate Strip to a Moving Line Load

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The response of an infinite plate strip under an arbitrarily distributed transverse moving line load is determined. The line of application of the load is perpendicular to the infinite edges, and the load is assumed to propagate parallel to the infinite edges of the plate at constant speed. The problem is formulated as a boundary value problem within the framework of the classical small-deflection theory of thin plates, and solutions are obtained in terms of trigonometric series. It is shown that the shape of the resulting deflection profile of the plate is strongly dependent upon the speed of propagation of the load as well as the magnitude of the damping coefficient. In the absence of damping, a denumerable infinity of critical speeds exists at which deflections become unbounded. In the presence of damping, however, deflections remain bounded.

Nomenclature

c = damping coefficient
c_{nc} = critical damping coefficient corresponding to nth harmonic component of load
D = $Eh^3/12(1 - \nu^2)$ = plate flexural rigidity
E = Young's modulus
h = plate thickness
l = width of infinite plate strip
m = plate mass per unit area
M_x, M_y = bending moments per unit length of sections of a plate perpendicular to x and y axes, respectively
M_{xy} = twisting moment per unit length of section of a plate perpendicular to x axis
p = intensity of line load
Q_x, Q_y = shearing forces parallel to z axis per unit length of sections of a plate perpendicular to x and y axes, respectively
t = time
v = speed of propagation of load
v_{nc} = critical speed corresponding to nth harmonic component of load
w = deflection of plate median surface
x, y = Cartesian coordinates of plate
ϵ_n = c/c_{nc} = damping ratio corresponding to nth harmonic component of load
ξ = $\pi(x - vt)/l$, η = $\pi y/l$ = dimensionless, moving Cartesian coordinates of plate
θ_n = v/v_{nc} = speed ratio corresponding to nth harmonic component of load
ν = Poisson's ratio
Ω_n = dimensionless wavelength/$(2\pi/n)$ = wavelength ratio
$\nabla^2 w$ = $(\partial^2 w/\partial x^2) + (\partial^2 w/\partial y^2)$
$\nabla^4 w$ = $(\partial^4 w/\partial x^4) + 2(\partial^4 w/\partial x^2 \partial y^2) + (\partial^4 w/\partial y^4)$

Introduction

WHEN a plate is subjected to transverse loads of constant intensity which move parallel to the surface of the plate, then the stresses induced in the plate are dependent not only upon the magnitude of the loads, but also strongly upon their speed of propagation. This phenomenon has already been investigated for simply supported, rectangular plates in Refs. 1 and 2, and for the case of a simply supported rectangular plate resting on an elastic foundation in Ref. 3. Livesley in Ref. 4 considers the response of an infinite plate on an elastic foundation to a traveling load. In these four cases critical speeds of propagation of the load are shown to exist, and the effect of damping is neglected. Thus deflections become unbounded when the load propagates with a speed equal to a critical speed. In addition, the solution presented in Ref. 4 is restricted to subcritical speeds.

One of the factors which complicate the phenomenon of the response of plates to traveling loads is the reflection of flexural waves from the boundaries of the plate normal to the direction of propagation of the load. This complication may be removed by considering solutions for plates which are infinite in the direction of propagation of the load. Thus it was with the hope of obtaining better insight into the mechanism of forced, flexural wave motion in plates that the present study was carried out.

The basic theory of plates is given in Refs. 5 and 6, which deal with static considerations. Transverse vibrations of plates are treated in Refs. 7–10, but none of these references are concerned with moving loads on plates. It is the purpose of this investigation to predict the response of a thin, simply supported, infinite plate strip under the action of a transverse, moving line load. The line of application of the load is taken perpendicular to the infinite edges, and the load is assumed to propagate parallel to the infinite edges of the plate at constant speed.

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Formulation and Solution

According to the classical small-deflection theory, the transverse deflections of the median plane of the plate are characterized by

$$D\nabla^4 w = \bar{q} \tag{1}$$

where $\bar{q}$ and D are the load intensity and flexural rigidity, respectively. The moments and shears are related to the deflections w by means of the following equations:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{2}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{3}$$

$$M_{xy} = -M_{yx} = -D(1 - \nu)\frac{\partial^2 w}{\partial x \, \partial y} \tag{4}$$

$$Q_x = -D\frac{\partial}{\partial x}(\nabla^2 w) \tag{5}$$

$$Q_y = -D\frac{\partial}{\partial y}(\nabla^2 w) \tag{6}$$

With particular reference to the dynamical nature of the problem at hand, we assume that the load intensity $\bar{q}$ is obtained by the superposition of three distinct quantities: (a) directly applied transverse forces of intensity q, (b) translational inertia forces of intensity $-m(\partial^2 w/\partial t^2)$, where m is the mass per unit area of plate, and (c) viscous damping forces of intensity $-c(\partial w/\partial t)$, where c is the damping coefficient. Thus Eq. (1) becomes

$$D\nabla^4 w + m\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = q \qquad (7)$$

In the derivation of Eq. (7) we have neglected the effect of rotatory inertia of the plate and the inertia of the load. In addition, all the assumptions of the classical theory of plates stated in Refs. 5 and 6 apply.

In what follows we shall be concerned with loads which propagate with constant speed v in the x-direction. It will be convenient to nondimensionalize the coordinates x and y, and to describe the response of the plate in a moving coordinate system. This is accomplished by the change of variables

$$\xi = \frac{\pi(x - vt)}{l}, \quad \eta = \frac{\pi y}{l} \qquad (8)$$

which transforms Eq. (7) into

$$\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} + \frac{mv^2 l^2}{\pi^2 D}\frac{\partial^2 w}{\partial \xi^2} - \frac{cvl^3}{\pi^3 D}\frac{\partial w}{\partial \xi} = \frac{l^4}{\pi^4 D} q \qquad (9)$$

The change of variables, Eqs. (8), may be given the following physical interpretation: An observer fixed with respect to the x-y coordinate system will see the distributed load q advance in the direction of the positive x-axis, and to him the deflection of the plate will appear to be dependent upon x, y, and t. However, an observer fixed with respect to the ξ,η coordinate system will move with the advancing load distribution, and to him the deflection surface will appear stationary—i.e., independent of t, and a function of ξ and η alone. We note that by neglecting damped transients due to the starting of the motion, we have made the implicit assumption that the load has been moving for a sufficiently long period. Thus we shall concentrate on the steady state dynamical process as characterized by Eq. (9).

We now consider the infinite plate strip $-\infty < \xi < \infty$, $0 \leqslant \eta \leqslant \pi$, where ξ and η are dimensionless moving coordinates as characterized by Eqs. (8). The moving, transverse line load is applied along the line $\xi = 0$, $0 \leqslant \eta \leqslant \pi$, and the motion is in the x-direction as implied by Eqs. (8) (see Fig. 1). The line load is assumed to be arbitrarily distributed and characterized by a finite trigonometric series or a Fourier series. In the analysis that follows we shall work with the nth harmonic component of the load and its corresponding deflection w_n, $n = 1,2,3,\ldots$. Any particular case to be considered subsequently will be obtained by superimposing the appropriate number of component solutions.

In the unloaded region of the plate $q = 0$, and according to Eq. (9), we require

$$\frac{\partial^4 w_n}{\partial \xi^4} + 2\frac{\partial^4 w_n}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w_n}{\partial \eta^4} + \frac{mv^2 l^2}{\pi^2 D}\frac{\partial^2 w_n}{\partial \xi^2} - \frac{cvl^3}{\pi^3 D}\frac{\partial w_n}{\partial \xi} = 0 \qquad (10)$$

The boundary conditions along the simply supported edges are given by

$$w_n(\xi,0) = 0, \qquad (\nabla^2 w_n)_{\xi,0} = 0$$
$$w_n(\xi,\pi) = 0, \qquad (\nabla^2 w_n)_{\xi,\pi} = 0 \qquad (11)$$

In addition, we require the deflection of the plate to remain bounded as $\xi \to \pm\infty$.

For convenience and in anticipation of future results we let

$$v_{n_c} = \frac{2\pi n}{l}\left(\frac{D}{m}\right)^{1/2}, \; \theta_n = \frac{v}{v_{n_c}} = \frac{vl}{2\pi n}\left(\frac{m}{D}\right)^{1/2}$$
$$c_{n_c} = \frac{4\pi^2 n^2}{l^2}(mD)^{1/2}, \; \epsilon_n = \frac{c}{c_{n_c}} = \frac{cl^2}{4\pi^2 n^2 (mD)^{1/2}} \qquad (12)$$

With these notations, Eqs. (12) becomes

$$\frac{\partial^4 w_n}{\partial \xi^4} + 2\frac{\partial^4 w_n}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w_n}{\partial \eta^4} + 4n^2\theta_n{}^2\frac{\partial^2 w_n}{\partial \xi^2} - 8n^3\theta_n\epsilon_n\frac{\partial w_n}{\partial \xi} = 0 \qquad (13)$$

In view of the boundary conditions (11), we assume a solution of the form

$$w_n(\xi,\eta) = \frac{K_n}{Q_n} f_n(\xi)\sin n\eta \qquad (14)$$

where K_n and Q_n are constants to be determined later. The form of the coefficient in Eq. (14) is chosen for convenience and in anticipation of future results. Upon substitution of

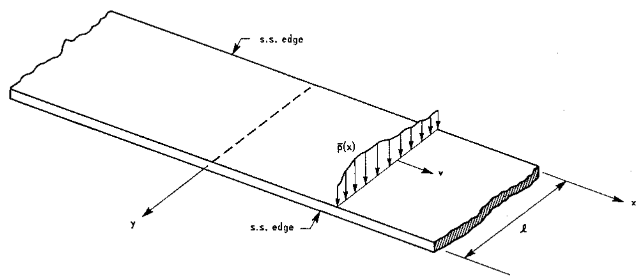

Fig. 1 Moving transverse line load

Eq. (14) into Eq. (13) we obtain the ordinary differential equation

$$\frac{d^4 f_n}{d\xi^4} + 2n^2(2\theta_n{}^2 - 1)\frac{d^2 f_n}{d\xi^2} - 8n^3\theta_n\epsilon_n\frac{df_n}{d\xi} + n^4 f_n = 0 \qquad (15)$$

Assuming $f_n(\xi) = e^{n\lambda\xi}$, where λ = constant, and substituting into Eq. (15), we obtain the characteristic equation

$$\lambda^4 + 2(2\theta_n{}^2 - 1)\lambda^2 - 8\theta_n\epsilon_n\lambda + 1 = 0 \qquad (16)$$

The solution of Eq. (16) depends upon the parameters ϵ_n and θ_n and is discussed from that point of view in the Appendix. For the present, we shall derive the solution when the roots are written as

$$\lambda_1 = -a + ib_1, \quad \lambda_2 = -a - ib_1$$
$$\lambda_3 = a + ib_2, \quad \lambda_4 = a - ib_2 \qquad (17)$$

where a, b_1, and b_2 are positive, real numbers. Equation (15) has four independent solutions. However, we must rule out solutions which become unbounded when $\xi \to \pm\infty$. Consequently, we may write the solution of Eq. (15) as

$$f_n{}^{(1)}(\xi) = e^{-na\xi}(C_1{}^{(1)}\cos nb_1\xi + C_2{}^{(1)}\sin nb_1\xi); \quad \xi \geq 0$$
$$f_n{}^{(2)}(\xi) = e^{na\xi}(C_1{}^{(2)}\cos nb_2\xi + C_2{}^{(2)}\sin nb_2\xi); \quad \xi \leq 0 \qquad (18)$$

where the superscripts 1 and 2 are attached to quantities pertaining to the region ahead and behind, respectively, of the moving load. The $C_i^{(j)}$, $i = 1,2$; $j = 1,2$ are constants of integration. The deflection, slope, and moment must be continuous under the load—i.e.,

$$w_n^{(1)}(0,\eta) = w_n^{(2)}(0,\eta) \tag{19}$$

$$\left(\frac{\partial w_n^{(1)}}{\partial \xi}\right)_{0,\eta} = \left(\frac{\partial w_n^{(2)}}{\partial \xi}\right)_{0,\eta} \tag{20}$$

$$M_{x_n}^{(1)}(0,\eta) = M_{x_n}^{(2)}(0,\eta) \tag{21}$$

The shear force is required to display the discontinuity

$$\lim_{\xi\to 0}[Q_{x_n}^{(2)}(\xi^-,\eta) - Q_{x_n}^{(1)}(\xi^+,\eta)] = pa_n \sin n\eta \tag{22}$$

where $pa_n \sin n\eta$ is the nth harmonic component of the line load. Upon substitution of the solution [(14) in conjunction with (18)] into the transition conditions (19) through (22), we obtain four linear algebraic equations in four unknowns. These equations, when solved, result in

$$C_1^{(1)} = C_1^{(2)} = a$$
$$C_2^{(1)} = \frac{b_2^2 - b_1^2 + 4a^2}{4b_1} \tag{23}$$
$$C_2^{(2)} = \frac{b_2^2 - b_1^2 - 4a^2}{4b_2}$$

where the quantities K_n and Q_n in Eq. (14) are taken as

$$K_n = \frac{pa_n l^3}{4\pi^3 n^3 D} \tag{24}$$

$$Q_n = 3a^4 + 2a^2(2\theta_n^2 - 1) + \theta_n^2(\theta_n^2 - 1) \tag{25}$$

Combining Eqs. (14), (18), and (23), we obtain the solution for $\xi \geqslant 0$:

$$w_n^{(1)}(\xi,\eta) = \frac{K_n}{Q_n} e^{-na\xi}\left[a \cos nb_1\xi + \frac{a^2 - (\theta_n\epsilon_n/a)}{b_1} \sin nb_1\xi\right] \sin n\eta = \frac{K_n}{b_1(Q_n)^{1/2}} e^{-na\xi}\cos(nb_1\xi - \varphi_n^{(1)}) \sin n\eta \tag{26}$$

where

$$\tan\varphi_n^{(1)} = \frac{[a^2 - (\theta_n\epsilon_n/a)]}{ab_1}$$
$$b_1 = \left(2\theta_n^2 - 1 + a^2 + \frac{2\theta_n\epsilon_n}{a}\right)^{1/2}$$

Similarly, for $\xi \leqslant 0$ we have the solution

$$w_n^{(2)}(\xi,\eta) = \frac{K_n}{Q_n} e^{na\xi}\left[a \cos nb_2\xi - \frac{a^2 + (\theta_n\epsilon_n/a)}{b_2} \sin nb_2\xi\right] \sin n\eta = \frac{K_n}{b^2(Q_n)^{1/2}} e^{na\xi}\cos(nb_2\xi - \varphi_n^{(2)})\sin n\eta \tag{27}$$

where

$$\tan\varphi_n^{(2)} = -\frac{[a^2 + (\theta_n\epsilon_n/a)]}{ab_2}$$
$$b_2 = \left(2\theta_n^2 - 1 + a^2 - \frac{2\theta_n\epsilon_n}{a}\right)^{1/2}$$

It is shown in the Appendix that b_2^2 is negative for certain combinations of θ_n and ϵ_n—i.e., when $\epsilon_n \geqslant \bar{\epsilon}_n$. In this case the solution (27) corresponding to $\xi \leqslant 0$ becomes complex. To write it in real form we observe that

$$(b_2^2)^{1/2} = i\,|b_2|, \qquad \sin ni|b_2|\,\xi = i \sinh n|b_2|\,\xi$$
$$\cos ni|b_2|\,\xi = \cosh n|b_2|\,\xi$$

and Eq. (27) becomes

$$w_n^{(2)}(\xi,\eta) = \frac{K_n}{Q_n} e^{na\xi}\left[a \cosh n|b_2|\,\xi - \frac{a^2 + (\theta_n\epsilon_n/a)}{|b_2|} \sinh n|b_2|\,\xi\right] \sin n\eta \tag{28}$$

When $\epsilon_n = \bar{\epsilon}_n$, $b_2^2 = 0$ (see the Appendix) and in this case Eq. (27) assumes the form

$$w_n^{(2)}(\xi,\eta) = \frac{K_n}{Q_n}\left[a - \left(a^2 + \frac{\theta_n\bar{\epsilon}_n}{a}\right) n\xi\right] e^{na\xi} \sin n\eta \tag{29}$$

For the particular case of zero damping $\epsilon_n = 0$ and $\theta_n \leq 1$ we have $a = (1 - \theta_n^2)^{1/2}$, $b_1 = b_2 = \theta_n$ (see Appendix), and $Q_n = 1 - \theta_n^2$. Therefore

$$C_1^{(1)} = C_1^{(2)} = (1 - \theta_n^2)^{1/2}$$
$$C_2^{(1)} = -C_2^{(2)} = \frac{1 - \theta_n^2}{\theta_n}$$

and, when $\xi \geq 0$, we have the solution

$$w_n^{(1)}(\xi,\eta) = K_n\cdot\left[\frac{1}{(1 - \theta_n^2)^{1/2}} \cos n\theta_n\xi + \frac{1}{\theta_n} \sin n\,\theta_n\xi\right] \sin n\eta\cdot \exp[-n(1 - \theta_n^2)^{1/2}\xi] = \frac{K_n}{\theta_n(1 - \theta_n^2)^{1/2}} \cos(n\theta_n\xi - \varphi_n)\sin n\eta \exp[-n(1 - \theta_n^2)^{1/2}\xi] \tag{30}$$

where

$$\tan\varphi_n = \frac{(1 - \theta_n^2)^{1/2}}{\theta_n}$$

and when $\xi \leqslant 0$, we have

$$w_n^{(2)}(\xi,\eta) = K_n\left[\frac{1}{(1 - \theta_n^2)^{1/2}} \cos n\theta_n\xi - \frac{1}{\theta_n} \sin n\theta_n\xi\right] \times \sin n\eta \exp[n(1 - \theta_n^2)^{1/2}\xi] = \frac{K_n}{\theta_n(1 - \theta_n^2)^{1/2}} \cos(n\theta_n\xi + \varphi_n)\sin n\eta \exp[n(1 - \theta_n^2)^{1/2}\xi] \tag{31}$$

For the particular case of zero damping $\epsilon_n = 0$ and $\theta_n > 1$, we have $a = 0$, $b_1 = \theta_n + (\theta_n^2 - 1)^{1/2}$, $b_2 = \theta_n - (\theta_n^2 - 1)^{1/2}$, and $Q_n = \theta_n^2(\theta_n^2 - 1)$, as shown in the Appendix. Therefore

$$C_1^{(1)} = C_1^{(2)} = 0$$
$$C_2^{(1)} = \frac{-\theta_n(\theta_n^2 - 1)^{1/2}}{\theta_n + (\theta_n^2 - 1)^{1/2}}$$
$$C_2^{(2)} = \frac{-\theta_n(\theta_n^2 - 1)^{1/2}}{\theta_n - (\theta_n^2 - 1)^{1/2}}$$

and when $\xi \geqslant 0$, the solution is

$$w_n^{(1)}(\xi,\eta) = \frac{-K_n}{\theta_n(\theta_n^2 - 1)^{1/2}[\theta_n + (\theta_n^2 - 1)^{1/2}]}\cdot \sin n[\theta_n + (\theta_n^2 - 1)^{1/2}]\xi\cdot\sin n\eta \tag{32}$$

For $\xi \leqslant 0$

$$w_n^{(2)}(\xi,\eta) = \frac{-K_n}{\theta_n(\theta_n^2 - 1)^{1/2}[\theta_n - (\theta_n^2 - 1)^{1/2}]}\cdot \sin n[\theta_n - (\theta_n^2 - 1)^{1/2}]\xi\cdot\sin n\eta \tag{33}$$

The deflection function for the particular limiting case of a

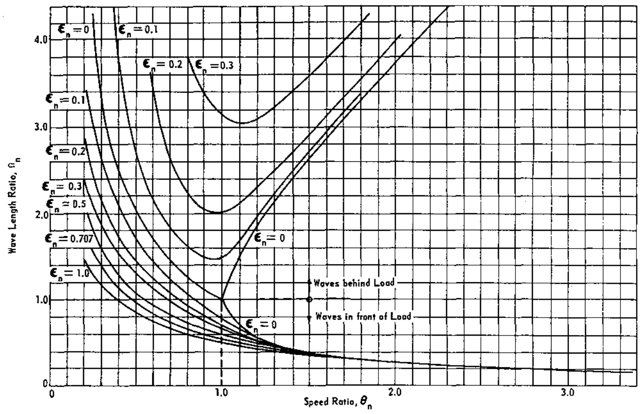

Fig. 2 Wave length ratio vs speed ratio

stationary load ($v = 0$) is obtained by taking the limit of Eqs. (30) and (31) as $\theta_n \to 0$:

$$w_n^{(1)}(\xi,\eta) = K_n e^{-n\xi}(1 + n\xi)\sin n\eta \quad (34)$$

$$w_n^{(2)}(\xi,\eta) = K_n e^{n\xi}(1 - n\xi)\sin n\eta \quad (35)$$

The static solution, Eqs. (34) and (35), is known and is given in Ref. 6.

The moments M_{xn} are now computed for the case $\epsilon_n = 0$ and $\theta_n < 1$ by substituting Eqs. (30) and (31) into Eq. (2). When $\xi > 0$, we obtain

$$M_{xn}^{(1)} = \frac{pa_n l}{4\pi n}\sin n\eta \cdot \left[\frac{(1+\nu)}{(1-\theta_n^2)^{1/2}}\cos n\theta_n\xi - \frac{(1-\nu)}{\theta_n}\sin n\theta_n\xi\right]\cdot$$

$$\exp[-n(1-\theta_n^2)^{1/2}\xi] = \frac{pa_n l}{4\pi n}\left[\frac{(1+\nu)^2}{1-\theta_n^2} + \frac{(1-\nu)^2}{\theta_n^2}\right]^{1/2}\cdot$$

$$\cos(n\theta_n\xi + \varphi_n)\cdot\sin n\eta\cdot\exp[-n(1-\theta_n^2)^{1/2}\xi] \quad (36)$$

where

$$\tan\varphi_n = \frac{1-\nu}{1+\nu}\frac{(1-\theta_n^2)^{1/2}}{\theta_n}$$

and for $\xi \leq 0$, we obtain

$$M_{xn}^{(2)} =$$

$$\frac{pa_n l}{4\pi n}\sin n\eta\left[\frac{(1+\nu)}{(1-\theta_n^2)^{1/2}}\cos n\theta_n\xi + \frac{(1-\nu)}{\theta_n}\sin n\theta_n\xi\right]\cdot$$

$$\exp[n(1-\theta_n^2)^{1/2}\xi] = \frac{pa_n l}{4\pi n}\left[\frac{(1+\nu)^2}{1-\theta_n^2} - \frac{(1-\nu)^2}{\theta_n^2}\right]^{1/2}\cdot$$

$$\cos(n\theta_n\xi - \varphi_n)\sin n\eta\cdot\exp[n(1-\theta_n^2)^{1/2}\xi] \quad (37)$$

where

$$\tan\varphi_n = \frac{1-\nu}{1+\nu}\frac{(1-\theta_n^2)^{1/2}}{\theta_n}$$

To obtain the moments M_{xn} for the case $\epsilon_n = 0$ and $\theta_n > 1$, we substitute Eqs. (32) and (33) into Eq. (2). For $\xi \geqslant 0$, we obtain

$$M_{xn}^{(1)} = -\frac{pa_n l}{4\pi n}\left\{\frac{[\theta_n + (\theta_n^2-1)^{1/2}]^2 + \nu}{\theta_n(\theta_n^2-1)^{1/2}[\theta_n + (\theta_n^2-1)^{1/2}]}\right\}\cdot$$

$$\sin n[\theta_n + (\theta_n^2-1)^{1/2}]\xi\cdot\sin n\eta \quad (38)$$

and when $\xi \leqslant 0$, we have

$$M_{xn}^{(2)} = -\frac{pa_n l}{4\pi n}\left\{\frac{[\theta_n - (\theta_n^2-1)^{1/2}]^2 + \nu}{\theta_n(\theta_n^2-1)^{1/2}[\theta_n - (\theta_n^2-1)^{1/2}]}\right\}\cdot$$

$$\sin n[\theta_n - (\theta_n^2-1)^{1/2}]\xi\cdot\sin n\eta \quad (39)$$

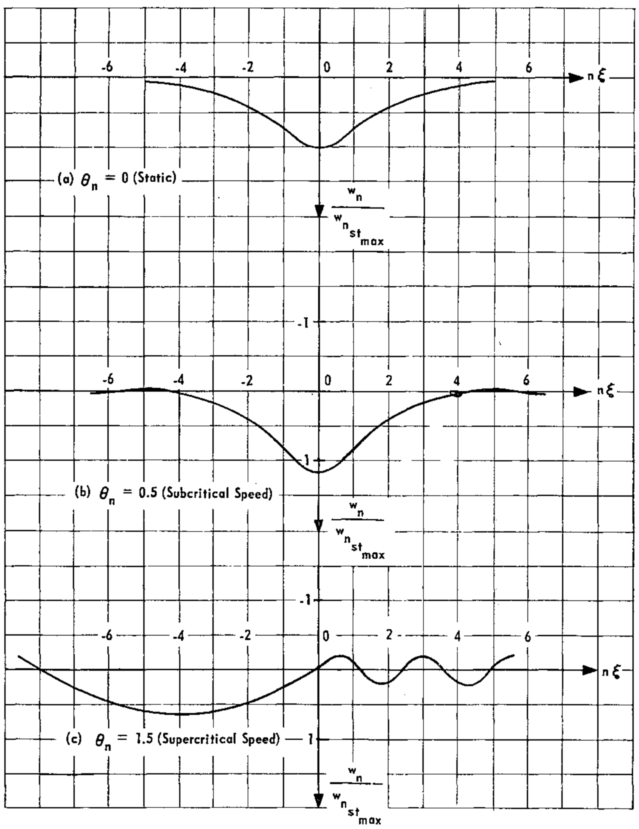

Fig. 3 Typical deflection profiles, line load, $\epsilon_n = 0$

The moment M_{xn} for the particular limiting case of a stationary load ($v = 0$) is obtained by taking the limit of Eqs. (36) and (37) as $\theta_n \to 0$

$$M_{xn}^{(1)} = -\frac{pa_n l}{4\pi n}e^{-n\xi}[(1-\nu)(1+n\xi) - 2]\sin n\eta,\ \xi \geqslant 0 \quad (40)$$

$$M_{xn}^{(2)} = -\frac{pa_n l}{4\pi n}e^{n\xi}[(1-\nu)(1-n\xi) - 2]\sin n\eta,\ \xi \leqslant 0 \quad (41)$$

These equations can also be obtained directly from the static solution, Eqs. (34) and (35).

Discussion of Results

Component Solutions

In the following discussion the term deflection profile refers to the trace of the intersection of any plane η = constant, $0 < \eta < \pi$ with the median plane of the plate as viewed in the plane η = constant. A study of the component solutions obtained reveals that the character of the deflection profile depends strongly upon the speed ratio θ_n and the damping ratio ϵ_n. Fig. 2 shows the variation of wavelength ratio Ω_n as a function of the speed ratio θ_n, with damping ratio ϵ_n as a parameter. The term wavelength ratio is defined as the ratio of the actual wavelength of the deflection profile to the (hypothetical) wavelength of the deflection profile at the critical speed in an undamped plate. With reference to Fig. 2, we note that for $\epsilon_n = 0$, he wavelengths in front and behind the load are identical for $\theta_n < 1$, but for $\theta_n > 1$ the wavelength in front of the load is smaller than the wavelength behind it. When damping is introduced ($\epsilon_n > 0$), we obtain different wave lengths in front and behind the load for all

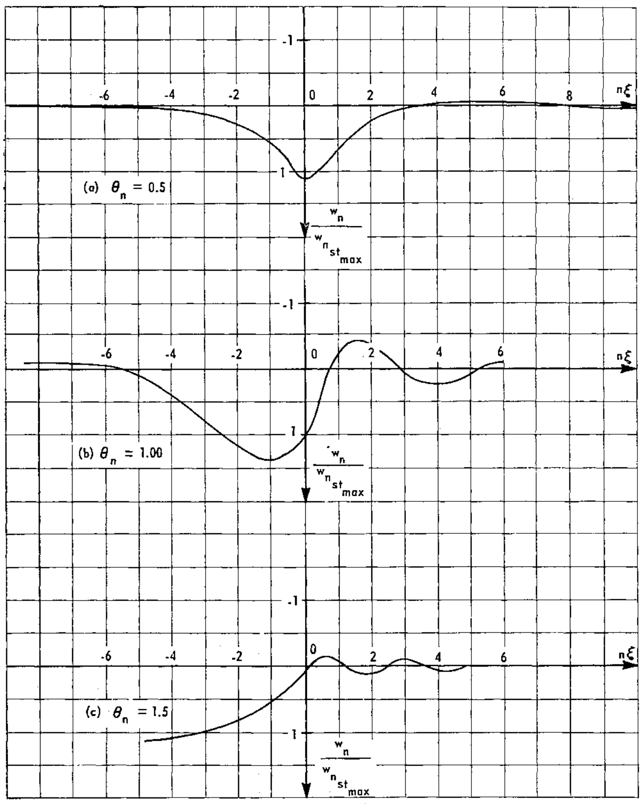

Fig. 4 Typical deflection profiles, line load, ϵ_n = 0.2

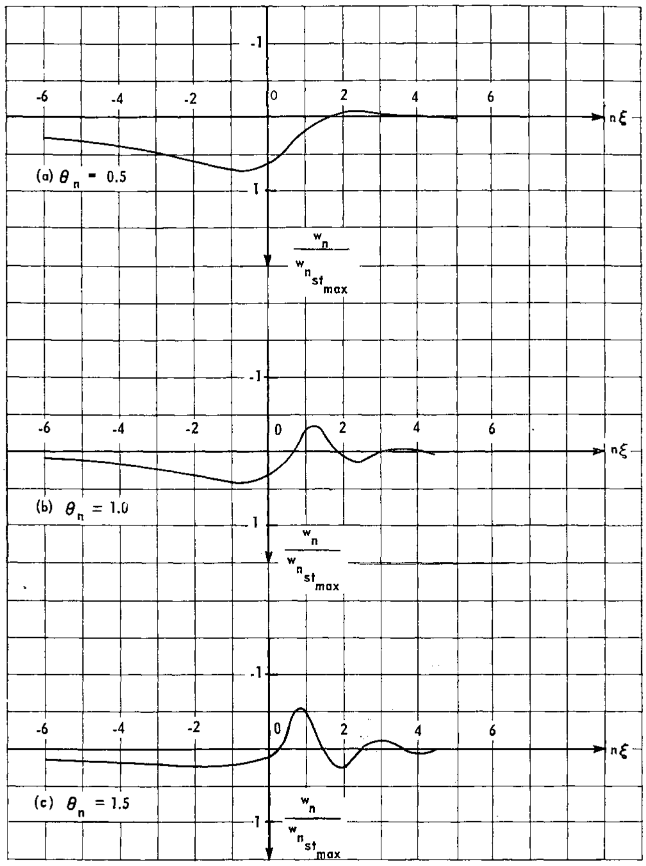

Fig. 5 Typical deflection profiles, line load, ϵ_n = 1.0

speed ratios θ_n, and for a given speed ratio, the wavelength in front of the load is always smaller than the wavelength behind the load. This difference increases substantially with an increase in θ_n for $\theta_n > 1$. We also note that as $\theta_n \to 0$, the wavelength becomes unbounded, and this result is confirmed by the known static solution.

There exist combinations of (θ_n,ϵ_n) which give rise to an "infinite wavelength" behind the load as demonstrated in the Appendix. For each speed ratio θ_n, there exists a value of $\epsilon_n = \bar{\epsilon}_n$ (denoted by the term "secondary critical damping") for which this is true. When $\epsilon_n \geqslant \bar{\epsilon}_n$ the wavelength behind the load is unbounded. Fig. 9 facilitates the determination of this phenomenon. If the point (θ_n,ϵ_n) lies on or above the curve in Fig. 9 (see Appendix), the wavelength behind the load will be unbounded. If the point (θ_n,ϵ_n) lies below the curve, we have an oscillatory deflection profile behind the load, and the wavelength is bounded. Because of the existence of an asymptote in Fig. 9, all component solutions will display infinite wavelength behind the load for all $\theta_n \geqslant 0$ when $\epsilon_n \geqslant 1/2$.

Some of the salient features of the deflection profiles obtained from the component solutions are presented in Figs. 3, 4, and 5 for a variety of combinations of speed ratio and damping ratio. The curves are normalized with respect to the maximum deflection under an identical static load. For the static case $\theta_n = 0$ (Fig. 3), the deflection is a maximum under the load and decreases monotonically to zero as $\xi \to \pm\infty$. The deflection profile is symmetrical with respect to the load point in this case. When $\epsilon_n = 0$, $0 \leqslant \theta_n < 1$ (subcritical speed), the maximum deflection still occurs under the load point, but the deflection profile is given by a damped sine wave symmetrical about the load point (see Fig. 3b). In the case of supercritical speed, $\theta_n > 1$, and zero damping, as shown in Fig. 3c, the deflection profile ceases to be symmetrical with respect to the point $\xi = 0$, and the amplitude of waves in front of the load is smaller than the amplitude of waves behind the load. In this case the deflection under the load is zero for all $\theta_n > 1$. We also note that for $\epsilon_n = 0$ and $\theta_n = 1$ (critical speed), deflections become unbounded.

When slight damping is introduced, $\bar{\epsilon}_n > \epsilon_n > 0$, the deflection profile is a damped sinusoid with amplitude decreasing exponentially with an increase in distance from the load point (see Fig. 4). In this case, deflections remain bounded for all $\theta_n \geqslant 0$. Three cases with damping greater than secondary-critical damping are shown in Fig. 5.

Fig. 6 is a plot of the dynamic amplification of deflection as a function of speed ratio θ_n, and Fig. 7 is the corresponding plot of dynamic amplification of moment M_{xn} as a function of speed ratio θ_n, both for the particular case of zero damping. Both graphs are normalized with respect to the corresponding maximum static values. With reference to these figures, we note that in the case of supercritical speed $\theta_n > 1$, the deflection amplification is greater behind the load than in front of the load, but the moment amplification is greater in front of the load than behind the load.

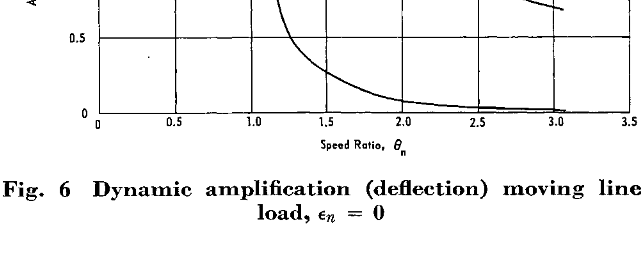

Fig. 6 Dynamic amplification (deflection) moving line load, ϵ_n = 0

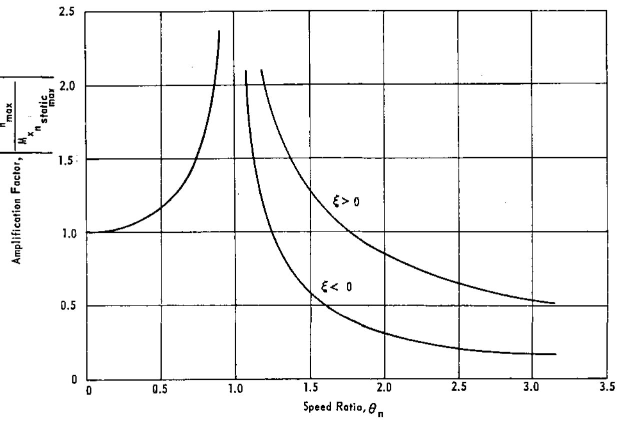

Fig. 7 Dynamic amplification (moment) moving line load, $\epsilon_n = 0$, $\nu = 0.3$

Superposition of Component Solutions

In most practical cases, the load distribution is characterized by either a finite trigonometric series or a Fourier series, and to obtain a solution it will be necessary to superimpose a finite or an infinite number of component solutions. Since the basic partial differential equation of motion is linear, and because each component solution satisfies the equation and the required boundary conditions, superposition is permissible and presents no special problems.

As an example, we consider the case of a moving, uniformly distributed line load of intensity p. In this case, the appropriate Fourier sine series expansion is

$$\bar{p} = \frac{4p}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin n\eta}{n}$$

so that $a_n = 4/\pi n$. If we are interested in the solution corresponding to $\xi \geqslant 0$, $\epsilon_n = 0$, then the applicable solution is given by the appropriate superposition of Eqs. (30) and (32):

$$\frac{\pi^4 D}{pl^3} w^{(1)}(\xi,\eta) = - \sum_{n=1,3,5,\ldots}^{N} \frac{\sin n[\theta_n + (\theta_n^2+1)^{1/2}]\xi \cdot \sin n\eta}{n^4\, \theta_n(\theta_n^2-1)^{1/2}[\theta_n + (\theta_n^2-1)^{1/2}]} + \sum_{n=N+2,N+4,\ldots}^{\infty} \left[\frac{\cos n\theta_n\xi}{(1-\theta_n^2)^{1/2}} + \frac{\sin n\theta_n\xi}{\theta_n}\right] \cdot \sin n\eta \cdot \frac{\exp[-n(1-\theta_n^2)^{1/2}\xi]}{n^4} \quad (42)$$

where $\theta_n > 1$ in the finite series and $\theta_n < 1$ in the infinite series of Eq. (42). Thus N is the greatest positive odd integer smaller than the quantity $(vl/2\pi)(m/D)^{1/2}$. We note that it is possible that $\theta_n < 1$ for all n. In this case the finite series in Eq. (42) is deleted, and only the infinite series applies.

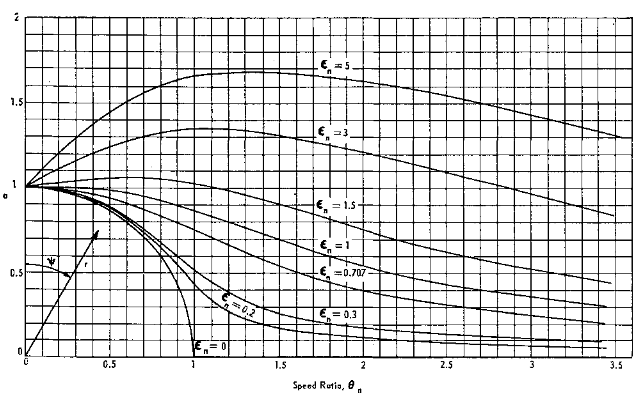

Fig. 8 Solution of characteristic equation (a vs θ_n, with ϵ_n as a parameter)

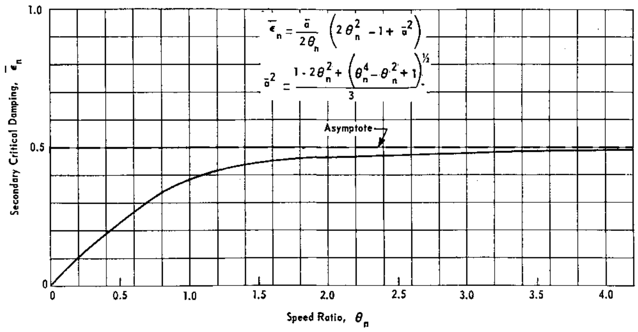

Fig. 9 Secondary critical damping vs speed ratio

Appendix: Roots of the Characteristic Equation

The characteristic equation of the fourth degree

$$\lambda^4 + 2(2\theta_n^2 - 1)\lambda^2 - 8\theta_n\epsilon_n\lambda + 1 = 0 \quad (43)$$

can always be written as the product of two quadratic factors with real coefficients

$$(\lambda^2 + 2a\lambda + \beta^2)\left(\lambda^2 - 2a\lambda + \frac{1}{\beta^2}\right) = 0 \quad (44)$$

(see, for instance, Ref. 11). The roots of Eq. (44) are

$$\lambda = -a \pm ib_1, \qquad \lambda = a \pm ib_2 \quad (45)$$

where

$$b_1^2 = \beta^2 - a^2, \qquad b_2^2 = \frac{1}{\beta^2} - a^2 \quad (46)$$

If, upon expanding Eq. (44), we equate coefficients of like powers of λ to those of Eq. (43) and solve for β^2 and $1/\beta^2$, we obtain

$$\beta^2 = 2\theta_n^2 - 1 + 2a^2 + \frac{2\theta_n\epsilon_n}{a}, \qquad \frac{1}{\beta^2} = 2\theta_n^2 - 1 + 2a^2 - \frac{2\theta_n\epsilon_n}{a} \quad (47)$$

Eliminating β from Eqs. (47), we obtain

$$a^6 + (2\theta_n^2 - 1)a^4 + \theta_n^2(\theta_n^2 - 1)a^2 - \theta_n^2\epsilon_n^2 = 0 \quad (48)$$

It is evident that Eq. (48) has at least one non-negative root a. Thus, we shall (arbitrarily) select $a \geqslant 0$ and compute the corresponding b_1 and b_2 from Eqs. (46) and (47)

$$b_1^2 = 2\theta_n^2 - 1 + a^2 + \frac{2\theta_n\epsilon_n}{a}, \qquad b_2^2 = 2\theta_n^2 - 1 + a^2 - \frac{2\theta_n\epsilon_n}{a} \quad (49)$$

[see Eqs. (46) and (47)].

We shall next investigate the dependence of a upon ϵ_n and θ_n. For this purpose we write Eq. (48) in the form

$$(\theta_n^2 + a^2)^2 - (\theta_n^2 + a^2) - \frac{\theta_n^2\epsilon_n^2}{a^2} = 0 \quad (50)$$

Solving for $(\theta_n^2 + a^2)$, we obtain

$$2(a^2 + \theta_n^2) = 1 + \left(1 + \frac{4\theta_n^2\epsilon_n^2}{a^2}\right)^{1/2} \quad (51)$$

and the positive square root applies because the quantity $2(a^2 + \theta_n^2)$ is non-negative. With reference to Fig. 8, let

$$\theta_n = r \sin\psi \qquad a = r \cos\psi \tag{52}$$

where $r \geqslant 0$, $0 \leqslant \psi \leqslant \pi/2$.

Introducing the change of variables (52) into Eq. (51), we obtain

$$2r^2 = 1 + (1 + 4\epsilon_n^2 \tan^2\psi)^{1/2} \tag{53}$$

This equation provides a convenient means for plotting (implicitly) a vs θ_n with $\epsilon_n > 0$ as a parameter, and this plot is shown in Fig. 8. We note that when $\epsilon_n \geqslant 0$, $r \geqslant 1$.

We must still discuss the case of zero damping. When $\epsilon_n = 0$ and $\theta_n \leqslant 1$, we have from Eq. (48):

$$a^4 + (2\theta_n^2 - 1)a^2 + \theta_n^2(\theta_n^2 - 1) = 0 \tag{54}$$

so that $a = (1 - \theta_n^2)^{1/2}$; $b_1 = b_2 = \pm\theta_n$, and, from Eq. (51), $r = 1$.

When $\epsilon_n = 0$ and $\theta_n > 1$, we have $a = 0$ as the required nonnegative root. Assuming a power series in ϵ_n for a, about $\epsilon_n = 0$, we have

$$a = \delta_1\epsilon_n + \delta_2\epsilon_n^2 + \ldots$$

and substituting this series in Eqs. (48), we obtain $a = \epsilon_n/(\theta_n^2 - 1)^{1/2}$ plus terms containing higher powers of ϵ_n. For sufficiently small ϵ_n and $\theta_n > 1$, we can take $a \cong \epsilon_n/(\theta_n^2 - 1)^{1/2}$.

Then Eq. (49) assumes the forms

$$b_1^2 \cong 2\theta_n^2 - 1 + \frac{\epsilon_n^2}{\theta_n^2 - 1} + 2\theta_n(\theta_n^2 - 1)^{1/2}$$
$$b_2^2 \cong 2\theta_n^2 - 1 + \frac{\epsilon_n^2}{\theta_n^2 - 1} - 2\theta_n(\theta_n^2 - 1)^{1/2} \tag{55}$$

Taking the limit of Eqs. (55) as $\epsilon_n \to 0$ we obtain

$$b_1 = \pm[\theta_n + (\theta_n^2 - 1)^{1/2}] \qquad b_2 = \pm[\theta_n - (\theta_n^2 - 1)^{1/2}] \tag{56}$$

We shall next investigate the sign of b_1^2 and b_2^2. If we introduce the transformation, Eqs. (52), into the first of Eqs. (49), we obtain

$$b_1^2 = r^2 - 1 + r^2 \sin^2\psi + 2\epsilon_n\tan\psi > r^2 - 1 \geqslant 0$$

because $r^2 \geqslant 1$. Hence, b_1^2 is nonnegative, and it is positive for $\theta_n > 0$, and vanishes for $\theta_n = 0$. However, the quantity b_2^2 may be positive, negative, or zero, and it is desirable to find the value of $\epsilon_n = \bar{\epsilon}_n$ as a function of θ_n for which $b_2^2 = 0$.

If we set $b_2^2 = 0$ in the second of Eq. (49) and eliminate ϵ_n from it with the help of Eq. (48) we obtain

$$3a^4 + 2a^2(2\theta_n^2 - 1) - 1 = 0 \tag{57}$$

The nonnegative, real root of Eq. (57) is

$$a = \bar{a} = \left[\frac{1 - 2\theta_n^2 + 2(\theta_n^4 - \theta_n^2 + 1)^{1/2}}{3}\right]^{1/2} \tag{58}$$

and therefore

$$\epsilon_n = \bar{\epsilon}_n = \frac{\bar{a}}{2\theta_n}(2\theta_n^2 - 1 + \bar{a}^2) = \frac{2\theta_n^2 - 1 + (\theta_n^4 - \theta_n^2 + 1)^{1/2}}{3\theta_n} \cdot \left[\frac{1 - 2\theta_n^2 + 2(\theta_n^4 - \theta_n^2 + 1)^{1/2}}{3}\right]^{1/2} \tag{59}$$

When $\theta_n = 0$, $\bar{a} = 1$, and $\bar{\epsilon}_n = 0$. When $\theta_n \to \infty$, $\bar{a} \to 1/(2\theta_n)$, and $\bar{\epsilon}_n \to 1/2$. A graph of Eq. (59) is shown in Fig. 9. When ϵ_n and θ_n satisfy (59), $b_2^2 = 0$. When $\epsilon_n < \bar{\epsilon}_n$, $b_2^2 > 0$, and when $\epsilon_n > \bar{\epsilon}_n$, $b_2^2 < 0$.

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